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Positive and Negative Hierarchies of Integrable Lattice Models Associated With a Hamiltonian Pair

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A difference Hamiltonian operator involving two arbitrary constants is presented, and it is used to construct a pair of nondegenerate Hamiltonian operators. The resulting Hamiltonian pair yields two difference hereditary operators, and the associated positive and negative hierarchies of nonlinear integrable lattice models are derived through the bi-Hamiltonian formulation. Moreover, the two lattice hierarchies are proved to have discrete zero curvature representations associated with a discrete spectral problem, which also shows that the positive and negative hierarchies correspond to positive and negative power expansions of Lax operators with respect to the spectral parameter, respectively. The use of zero curvature equation leads us to conclude that all resulting integrable lattice models are local and that the integrable lattice models in the positive hierarchy are of polynomial type and the integrable lattice models in the negative hierarchy are of rational type.

KEY WORDS: Integrable lattice model; Hamiltonian operator; bi-Hamiltonian formulation; hereditary recursion operator; zero curvature representation.

1. INTRODUCTION

It almost becomes a curiosity to find new nonlinear integrable models due to difficulty in the construction of integrable models. The related study will, however, provide clues for classifying integrable models. There are a few powerful techniques for generating nonlinear continuous integrable models, for example, Lax pair (Ablowitz and Clarkson, 1991; Das, 1989; Lax, 1968), recursion operator (Ma, 1998; Olver, 1977; Zakharov and Konopelchenko, 1984), bi-Hamiltonian formulation (Faddeev and Takhtajan, 1987; Magri, 1978; Tu, 1989), and R-matrix approach (Blaszak, 1998). Many continuous nonlinear integrable models such as the KdV and the KP equations have been systematically analyzed. There are also

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a couple of generalizations of the KdV equation presented recently (Antonowitz and Fordy, 1987; Gürses and Karasu, 1998; Ma, 1989; Ma and Pavlov, 1998). Although the techniques mentioned above also work well for discrete integrable models, there is not so much work done using the bi-Hamiltonian formulation to search for nonlinear discrete integrable models. One of the reasons is that the existence of bi-Hamiltonian structures may be too strong for many models integrable in other different senses. Nevertheless, the beauty hidden behind the bi-Hamiltonian formulation still drives one to fish in the lake of bi-Hamiltonian models.

In this paper, we present a difference Hamiltonian operator containing two arbitrary constants, and a pair of related nondegenerate Hamiltonian operators. This Hamiltonian pair leads to two hereditary operators, and the associated positive and negative hierarchies of nonlinear integrable lattice models are constructed by using the bi-Hamiltonian formulation. The resulting two hierarchies are all bi-Hamiltonian, and thus possess infinitely many commuting symmetries and infinitely many commuting conserved functionals. Moreover, both positive and negative hierarchies are proved to have zero curvature representations associated with a discrete spectral problem, which also shows that the positive and negative hierarchies correspond to positive and negative power expansions of Lax operators with respect to the spectral parameter, respectively. The use of zero curvature equation leads us to conclude that all resulting integrable lattice models are local, and that the positive hierarchy consists of integrable lattice models of polynomial type and the negative hierarchy consists of integrable lattice models of rational type. Finally, conclusions and some concluding remarks are given.

2. A HAMILTONIAN OPERATOR PAIR AND TWO LATTICE INTEGRABLE HIERARCHIES

We first briefly introduce some notation in the Hamiltonian theory of lattice models (see Oevel *et al.* (1989) and Ragnisco and Santini (1990) for more detailed information). We assume that $u = (r, s)^T$, where $r = r(n, t)$ and $s = s(n, t)$ are real functions defined over $Z \times R$. The shift operator E, its inverse, and two difference operators D and Δ are defined as follows

$$(Ef)(n) = f(n+1), \quad (E^{-1}f)(n) = f(n-1), \quad n \in Z, \tag{1a}$$

$$(Df)(n) = f(n+1) - f(n), \quad (\Delta f)(n) = f(n+1) - f(n-1), \quad n \in Z, \tag{1b}$$

where f is a lattice function, i.e., a function from Z to R. As normal, we write $f^{(k)} = E^k f$, $k \in Z$. The variational derivative, the Gateaux derivative and the inner product are defined by

$$\frac{\delta \tilde{H}}{\delta u} = \left(\frac{\delta \tilde{H}}{\delta r}, \frac{\delta \tilde{H}}{\delta s}\right)^T, \ \frac{\delta \tilde{H}}{\delta r} = \sum_{m \in Z} E^{-m}\left(\frac{\partial H}{\partial r^{(m)}}\right), \ \frac{\delta \tilde{H}}{\delta s} = \sum_{m \in Z} E^{-m}\left(\frac{\partial H}{\partial s^{(m)}}\right), \tag{2}$$

$$J'(u)[v] = \frac{\partial}{\partial \varepsilon} J(u + \varepsilon v)|_{\varepsilon=0}, \tag{3}$$

$$\langle f, g \rangle = \sum_{n \in Z} \langle f(n), g(n) \rangle. \tag{4}$$

Here $\tilde{H} = \sum_{n \in Z} H(u(n))$ is a functional; J is an operator, f, g, and v are two-dimensional vector functions; and $\langle f(n), g(n) \rangle$ denotes the standard inner product of $f(n)$ and $g(n)$ in the Euclidean space R^2.

Denote by J^* the adjoint operator of J with respect to (4), and thus we have $\langle f, J^* g \rangle = \langle Jf, g \rangle$. Obviously, the adjoint operator of E is E^{-1}. If an operator J has the property $J = -J^*$, then J is called to be skew-symmetric. The operator $\Delta = E - E^{-1}$ is skew-symmetric, and the inverse of Δ will only be defined on those lattice functions in the range of Δ and produces no additive constant. For example, if we have $f = (E - 1)g$ and $f|_{u=0} = 0$, then we set $g = (E - 1)^{-1} f$. In practice, this inverse can be explicitly realized (Ma and Fuchssteiner, 1999) as

$$\Delta^{-1} = \frac{1}{2} \left(\sum_{k=-\infty}^{-1} E^{2k+1} - \sum_{k=0}^{\infty} E^{2k+1} \right). \tag{5}$$

It is known that a linear operator J is Hamiltonian, if J is a skew-symmetric operator satisfying the Jacobi identity, i.e., it satisfies that

$$\langle f, Jg \rangle = -\langle Jf, g \rangle,$$
$$\langle J'(u)[Jf]g, h \rangle + \mathrm{Cycle}(f, g, h) = 0.$$

The associated Poisson bracket with a given Hamiltonian operator J is given by

$$\{\tilde{H}_1, \tilde{H}_2\}_J = \left\langle \frac{\delta \tilde{H}_1}{\delta u}, J \frac{\delta \tilde{H}_2}{\delta u} \right\rangle.$$

Now, motivated by the work of Ma and Zhou (1999), we introduce a specific 2×2 matrix local difference operator:

$$\begin{aligned} J(u) &= J(u; \alpha, \beta) \\ &= \begin{pmatrix} -\alpha r E r + \alpha r E^{-1} r & -\alpha r s + \alpha r E^{-1} s + \beta r - \beta r E^{-1} \\ \alpha s r - \alpha s E r + \beta E r - \beta r & \beta r E^{-1} - \beta E r \end{pmatrix}, \end{aligned} \tag{6}$$

where $u = (r, s)^T$ as before, and α and β are two arbitrary constants. The action of J is taken as the left multiplication, and thus it is linear (Tu and Ma, 1990). Note that J itself is nonlinear with respect to the dependent variable $u = (r, s)^T$.

Theorem 2.1. *The local difference matrix operator J defined by (6) is Hamiltonian for all values of two constants α and β.*

Proof:

(i) A straightforward computation gives $J^* = -J$, and thus J is a skew-symmetric operator.
(ii) The proof of the Jacobi identity $\langle J'(u)[Jf]g, h\rangle + \mathrm{Cycle}(f, g, h) = 0$ is given in the Appendix.

Therefore, J is Hamiltonian whatever α and β are. The proof of the theorem is finished. □

Two specific cases of the constants α and β are interesting. The case of $\alpha = 0$ and $\beta = 1$ leads to the Hamiltonian operator:

$$J_1 = \begin{pmatrix} 0 & r - rE^{-1} \\ -r + Er & rE^{-1} - Er \end{pmatrix}, \tag{7a}$$

and the other case of $\alpha = 1$ and $\beta = 0$ leads to the Hamiltonian operator:

$$J_2 = \begin{pmatrix} -rEr + rE^{-1}r & -rs + rE^{-1}s \\ sr - sEr & 0 \end{pmatrix}. \tag{7b}$$

By Theorem 1, the sum of J_1 and J_2 is also Hamiltonian, and thus J_1 and J_2 constitute a pair of Hamiltonian operators.

Both of the two operators J_1 and J_2 are nondegenerate. Namely, if there is a 1×2 matrix local difference operator M_i such that $M_i J_i = 0$, then $M_i = 0$, where $i = 1$ or $i = 2$. These are easy to prove. Let us look at the situation for J_1. Suppose that we have $M_1 = (P, Q)$ such that $M_1 J_1 = 0$, where P and Q are two local difference operators. Then, we have $Q(-r + Er) = 0$, which can lead to $Q = 0$, and further $P(r - rE^{-1}) = 0$, which will lead to $P = 0$. Therefore, we have $M_1 = 0$. This implies that J_1 is nondegenerate. The proof for the nondegenerency of J_2 is completely similar and so we omit it. Actually, over some well-selected spaces of vector lattice functions (e.g., we will see that we can have the space spanned by the positive hierarchy for J_1^{-1} and the space spanned by the negative hierarchy for J_2^{-1}), the inverse operators of these two Hamiltonian operators J_1 and J_2 can be explicitly expressed as follows:

$$J_1^{-1} = \begin{pmatrix} (1 - E^{-1})^{-1}\dfrac{1}{r} + \dfrac{1}{r}(E - 1)^{-1} & \dfrac{1}{r}(E - 1)^{-1} \\ (1 - E^{-1})^{-1}\dfrac{1}{r} & 0 \end{pmatrix}, \tag{8a}$$

and

$$J_2^{-1} = \begin{pmatrix} 0 & -\dfrac{1}{r}(E - 1)^{-1}\dfrac{1}{s} \\ -\dfrac{1}{s}(1 - E^{-1})^{-1}\dfrac{1}{r} & \dfrac{1}{s}(1 + E)(E - 1)^{-1}\dfrac{1}{s} \end{pmatrix}, \tag{8b}$$

where $(E-1)^{-1} = (1+E^{-1})\Delta^{-1}$ and $(1-E^{-1})^{-1} = (E+1)\Delta^{-1}$. Using (5), the inverse operators $(E-1)^{-1}$ and $(1-E^{-1})^{-1}$ can have the expressions:

$$(E-1)^{-1} = (1+E^{-1})\Delta^{-1} = \frac{1}{2}\left(\sum_{k=-\infty}^{-1} E^k - \sum_{k=0}^{\infty} E^k\right), \tag{9a}$$

$$(1-E^{-1})^{-1} = (E+1)\Delta^{-1} = \frac{1}{2}\left(\sum_{k=-\infty}^{-1} E^{k+1} - \sum_{k=0}^{\infty} E^{k+1}\right). \tag{9b}$$

Let us now introduce

$$\Psi = J_1^{-1} J_2, \quad \Psi^{-1} = J_2^{-1} J_1, \tag{10}$$

where J_1 and J_2 and their inverses are defined by (7) and (8), respectively. Then by the bi-Hamiltonian theory (Fuchssteiner and Fokas, 1981/82), their adjoint operators

$$\Phi := \Psi^* = \begin{pmatrix} -r(1+E^{-1}) - r[(E-E^{-1})r + (1-E^{-1})s](1-E^{-1})^{-1}\frac{1}{r} & -r(1+E^{-1}) \\ -s - s(E-1)r(1-E^{-1})^{-1}\frac{1}{r} & -s \end{pmatrix} \tag{11}$$

and

$$\Phi^{-1} := (\Psi^{-1})^* = \begin{pmatrix} -r(1-E^{-1})\frac{1}{s}(1-E^{-1})^{-1}\frac{1}{r} & -(rE^{-1}-r)\frac{1}{s}(E+1)(E-1)^{-1}\frac{1}{s} \\ -(rE^{-1}-Er)\frac{1}{s}(1-E^{-1})^{-1}\frac{1}{r} & -\frac{1}{s} + (rE^{-1}-Er)\frac{1}{s}(E+1)(E-1)^{-1}\frac{1}{s} \end{pmatrix} \tag{12}$$

are two hereditary operators. We will show that they yield two hierarchies of local integrable lattice models, although they themselves are nonlocal operators.

On the other hand, we can have the following bi-Hamiltonian initial equalities:

$$J_1\frac{\delta \tilde{H}_1}{\delta u} = J_2\frac{\delta \tilde{H}_0}{\delta u},\ J_2\frac{\delta \tilde{G}_1}{\delta u} = J_1\frac{\delta \tilde{G}_0}{\delta u}, \tag{13}$$

more precisely,

$$\frac{\delta \tilde{H}_1}{\delta u} = \Psi\frac{\delta \tilde{H}_0}{\delta u},\ \frac{\delta \tilde{G}_1}{\delta u} = \Psi^{-1}\frac{\delta \tilde{G}_0}{\delta u},$$

where four Hamiltonian functionals are given by $\tilde{H}_i = \Sigma_{n\in Z} H_i(u(n))$, $\tilde{G}_i = \Sigma_{n\in Z} G_i(u(n))$, $0 \le i \le 1$, and

$$H_0 = s + \frac{r}{2} + \frac{r^{(1)}}{2}, \tag{14a}$$

$$H_1 = -\frac{1}{4}\left(2s^2 + 4rs + 4r^{(1)}s + 2r^{(1)}r + rr^{(-1)} + r^2 + r^{(2)}r^{(1)} + \left(r^{(1)}\right)^2\right), \tag{14b}$$

$$G_0 = \frac{1}{2}\,\mathrm{In}\, r - \mathrm{In}\, s, \quad G_1 = -\frac{1}{s} - \frac{r}{ss^{(-1)}}. \tag{15}$$

Now, we can introduce two hierarchies of nonlinear integrable lattice models as follows:

$$u_{t_m} = X_m := J_1\Psi^m f_0, \quad f_0 := \frac{\delta \tilde{H}_0}{\delta u} = (1, 1)^T, \ \ m \ge 0, \tag{16}$$

and

$$u_{t_m} = Y_m := J_2\Psi^{-m} g_0, \quad g_0 := \frac{\delta \tilde{G}_0}{\delta u} = \left(\frac{1}{2r}, \frac{-1}{s}\right)^T, \ \ m \ge 0. \tag{17}$$

Obviously, we have the recursion structures

$$J_1\Psi^{m+1} f_0 = J_2\Psi^m f_0, \quad J_2\Psi^{-(m+1)} g_0 = J_1\Psi^{-m} g_0, \quad m \ge 0.$$

Note that J_1 and J_2 form a Hamiltonian pair, both of which are nondegenerate; and that f_0, Ψf_0, g_0, and $\Psi^{-1} g_0$ are gradient. Then based on the bi-Hamiltonian theory [especially on Lemma 7.25 (Olver, 1986)], all vector lattice functions $\Psi^m f_0$ and $\Psi^{-m} g_0$, $m \ge 0$, are gradient. Therefore, all lattice models with $m \ge 1$ in the two hierarchies (16) and (17) possess the following bi-Hamiltonian structures

$$u_{t_m} = X_m = J_1\frac{\delta \tilde{H}_m}{\delta u} = J_2\frac{\delta \tilde{H}_{m-1}}{\delta u}, \quad m \ge 1, \tag{18}$$

and

$$u_{t_m} = Y_m = J_2\frac{\delta \tilde{G}_m}{\delta u} = J_1\frac{\delta \tilde{G}_{m-1}}{\delta u}, \quad m \ge 1, \tag{19}$$

where the Hamiltonian functionals $\tilde{H}_m$ and $\tilde{G}_m$ are given by

$$\tilde{H}_m = \sum_{n\in Z} H_m(u(n)), \quad H_m := \int_0^1 u^T(\Psi^m f_0)(\mu u)\, d\mu, \quad m \ge 0, \tag{20}$$

and

$$\tilde{G}_m = \sum_{n\in Z} G_m(u(n)), \quad G_m := \int_0^1 u^T(\Psi^{-m} g_0)(\mu u)\, d\mu, \quad m \ge 0. \tag{21}$$

Two first nontrivial lattice models in the positive hierarchy (16) and the negative hierarchy (17) read as

$$r_{t_1} = r\left(s^{(-1)} - s\right) + r\left(r^{(-1)} - r^{(1)}\right), \quad s_{t_1} = rs - r^{(1)}s;$$

and

$$r_{t_1} = \frac{r}{s^{(-1)}} - \frac{r}{s}, \quad s_{t_1} = \frac{r^{(1)}}{s^{(1)}} - \frac{r}{s^{(-1)}};$$

respectively. Note that these two lattice models are all local. The first one is polynomial and the second one is rational in u and its shifts.

It is known that if J is a Hamiltonian operator, then

$$\left[J\frac{\delta \tilde{H}_1}{\delta u}, J\frac{\delta \tilde{H}_2}{\delta u}\right] = J\frac{\delta\{\tilde{H}_1, \tilde{H}_2\}_J}{\delta u},$$

where the commutator is defined by

$$[X, Y] := \frac{\partial}{\partial \varepsilon}(X(u + \varepsilon Y) - Y(u + \varepsilon X))|_{\varepsilon=0}.$$

Thus, for our two hierarchies of lattice models, we have

$$[X_m, X_1] = \left[J_1\frac{\delta \tilde{H}_m}{\delta u}, J_1\frac{\delta \tilde{H}_1}{\delta u}\right] = J_1\frac{\delta\{\tilde{H}_m, \tilde{H}_1\}_{J_1}}{\delta u} = 0, \quad m, l \geq 0,$$

and

$$[Y_m, Y_1] = \left[J_2\frac{\delta \tilde{G}_m}{\delta u}, J_2\frac{\delta \tilde{G}_1}{\delta u}\right] = J_2\frac{\delta\{\tilde{G}_m, \tilde{G}_1\}_{J_2}}{\delta u} = 0, \quad m, l \geq 0,$$

where the commutativity of the Hamiltonian functionals is a consequence of the recursion relation

$$\frac{\delta \tilde{H}_{m+1}}{\delta u} = \Psi\frac{\delta \tilde{H}_m}{\delta u}, \frac{\delta \tilde{G}_{m+1}}{\delta u} = \Psi^{-1}\frac{\delta \tilde{G}_m}{\delta u}, \quad m \geq 0.$$

Therefore, $\{X_m\}_{m=0}^{\infty}$ and $\{\tilde{H}_m\}_{m=0}^{\infty}$ are infinitely many commuting symmetries and infinitely many commuting conserved functionals of the positive lattice hierarchy (16), and $\{Y_m\}_{m=0}^{\infty}$ and $\{\tilde{G}_m\}_{m=0}^{\infty}$ are infinitely many commuting symmetries and infinitely many commuting conserved functionals of the negative lattice hierarchy (17).

3. A DISCRETE SPECTRAL PROBLEM AND ZERO CURVATURE REPRESENTATIONS

What we want to present now is zero curvature representations for the resulting positive and negative lattice hierarchies (16) and (17), which will also show that the positive and negative hierarchies correspond to positive and negative power expansions of Lax operators with respect to the spectral parameter, respectively. The use of zero curvature equation leads us to conclude that all resulting integrable lattice models are local, and that the positive hierarchy consists of integrable lattice models of polynomial type and the negative hierarchy consists of integrable lattice models of rational type.

To proceed, let us introduce the following discrete spectral problem

$$E\varphi = U(u,\lambda)\varphi, \quad U(u,\lambda) = \begin{pmatrix} 0 & 1 \\ r & \lambda + \dfrac{s}{\lambda} \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{22}$$

where $u = (r, s)^T$ as before. This discrete spectral problem is equivalent to

$$(E - rE^{-1} - \lambda^{-1}s)\psi = \lambda\psi, \quad \psi = E\varphi_1.$$

To get the associated integrable lattice models, we first solve the stationary discrete zero curvature equation

$$(E\Gamma_1)U - U\Gamma_1 = 0. \tag{23}$$

Upon setting

$$\Gamma_1 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

we find that Eq. (23) becomes

$$\begin{aligned} & rb^{(1)} - c = 0, \\ & \left(a + a^{(1)}\right) + b^{(1)}\lambda + \frac{s}{\lambda}b^{(1)} = 0, \\ & c^{(1)} - rb - \lambda\left(a^{(1)} - a\right) - \frac{s}{\lambda}\left(a^{(1)} - a\right) = 0. \end{aligned} \tag{24}$$

The substitution of

$$a = \sum_{m=0}^{\infty} a_m\lambda^{-2m}, \quad b = \sum_{m=0}^{\infty} b_m\lambda^{-2m+1}, \quad c = \sum_{m=0}^{\infty} c_m\lambda^{-2m+1}$$

into (24) leads to the initial relation:

$$b_0^{(1)} = 0, \quad c_0 = 0, \quad a_0 - a_0^{(1)} = -c_0^{(1)} + rb_0,$$

and the recursion relation:

$$\begin{aligned} & rb_m^{(1)} - c_m = 0, \quad m \geq 0, \\ & b_{m+1}^{(1)} + sb_m^{(1)} + \left(a_m + a_m^{(1)}\right) = 0, \quad m \geq 0, \\ & \left(a_{m+1}^{(1)} - a_{m+1}\right) + s\left(a_m^{(1)} - a_m\right) + rb_{m+1} - c_{m+1}^{(1)} = 0, \quad m \geq 0. \end{aligned} \tag{25}$$

We choose the initial data satisfying the above initial relation

$$a_0 = -\frac{1}{2}, \quad b_0 = 0.$$

Then, the recursion relation (25) uniquely determines the lattice functions $a_m, b_m, c_m; m \geq 1$, and the first few lattice functions are given by

$$\begin{aligned} &a_1 = r, \quad b_1 = 1, \quad c_1 = r, \\ &a_2 = -r^{(1)}r - rr^{(-1)} - r^2 - rs - rs^{(-1)}, \\ &b_2 = -r - r^{(-1)} - s^{(-1)}, \quad c_2 = -rs - r^2 - rr^{(1)}. \end{aligned}$$

Moreover, from (23), we can know (Tu, 1990) that $(E-1)\mathrm{tr}(\Gamma_1^k) = 0$ for all $k \geq 1$. In particular, we have $\mathrm{tr}(\Gamma_1^2) = 2(a^2 + bc)$ is a constant, and let us say γ_1. Then, we obtain a recursion relation for a_m:

$$a_{m+1} = \sum_{i=1}^{m} a_i a_{m-i+1} + \sum_{i=1}^{m+1} b_i c_{m-i+2} - \frac{1}{2}\gamma_1, \quad m \geq 1.$$

This relation, together with the first two recursion relations in (25), implies through the mathematical induction that all lattice functions $a_m, b_m, c_m; m \geq 1$, are local, and they are just difference polynomials in the two dependent variables r and s.

Now we define

$$V_m = (\lambda^{2m}\Gamma_1)_+ \equiv \begin{pmatrix} \sum_{i=0}^{m} a_i\lambda^{2m-2i} & \sum_{i=0}^{m} b_i\lambda^{2m-2i+1} \\ \sum_{i=0}^{m} c_i\lambda^{2m-2i+1} & -\sum_{i=0}^{m} a_i\lambda^{2m-2i} \end{pmatrix}, \quad m \geq 0, \tag{26}$$

and then we can obtain

$$E(V_m)U - UV_m = \begin{pmatrix} 0 & -b_{m+1}^{(1)} \\ c_{m+1} & \dfrac{-s\left(a_m^{(1)} - a_m\right)}{\lambda} \end{pmatrix}.$$

To present the associated hierarchy of lattice models, we take a modification

$$\Delta_m = \begin{pmatrix} b_{m+1} & 0 \\ 0 & 0 \end{pmatrix},$$

and define the auxiliary Lax operators

$$V^{[m]} = V_m + \Delta_m, \quad m \geq 0. \tag{27}$$

Through a direct calculation, we can have

$$\left(EV^{[m]}\right)U - UV^{[m]} = \begin{pmatrix} 0 & 0 \\ c_{m+1} - rb_{m+1} & \dfrac{-s\left(a_m^{(1)} - a_m\right)}{\lambda} \end{pmatrix}.$$

which is consistent with U_{t_m}. Then for all $m \geq 0$, we introduce the following auxiliary spectral problems associated with the spectral problem (22):

$$\varphi_{t_m} = V^{[m]}\varphi, \quad m \geq 0. \tag{28}$$

The compatibility conditions of Eqs. (22) and (28) are

$$U_{t_m} = \left(EV^{[m]}\right)U - UV^{[m]}, \quad m \geq 0, \tag{29}$$

which give rise to the following hierarchy of lattice models

$$r_{t_m} = c_{m+1} - rb_{m+1}, \quad m \geq 0, \tag{30a}$$

$$s_{t_m} = -s\left(a_m^{(1)} - a_m\right), \quad m \geq 0. \tag{30b}$$

These models can be rewritten as

$$u_{t_m} = \begin{pmatrix} r \\ s \end{pmatrix}_{t_m} = J_1 \begin{pmatrix} \dfrac{a_{m+1}}{r} \\ \dfrac{c_{m+1}}{r} \end{pmatrix}, \quad m \geq 0, \tag{31}$$

where J_1 is defined by (7a). Obviously, it follows from the recursion relation for the lattice functions a_m, b_m, and c_m that the following recursion relation

$$\begin{pmatrix} \dfrac{a_{m+1}}{r} \\ \dfrac{c_{m+1}}{r} \end{pmatrix} = \Psi \begin{pmatrix} \dfrac{a_m}{r} \\ \dfrac{c_m}{r} \end{pmatrix}, \quad m \geq 1$$

holds. Therefore, we have

$$J_1 \begin{pmatrix} \dfrac{a_{m+1}}{r} \\ \dfrac{c_{m+1}}{r} \end{pmatrix} = J_1 \Psi^m \begin{pmatrix} \dfrac{a_1}{r} \\ \dfrac{c_1}{r} \end{pmatrix} = J_1 \Psi^m f_0 = X_m, \quad m \geq 0.$$

and so the lattice models (31) are just the positive lattice hierarchy (16). This implies that the positive lattice hierarchy (16) is local and it has the discrete zero curvature representations (29). Moreover, the Lax operators $V^{[m]}$ of the positive hierarchy (16) only have positive powers of the spectral parameter λ.

In order to present the discrete zero curvature representations for the negative lattice hierarchy (17), we consider the stationary discrete zero curvature equation

$$(E\Gamma_2)U - U\Gamma_2 = 0, \tag{32}$$

where $\Gamma_2 = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}$ with A, B, and C being chosen as

$$A = \sum_{m=0}^{\infty} A_m \lambda^{2m}, \quad B = \sum_{m=0}^{\infty} B_m \lambda^{2m-1}, \quad C = \sum_{m=0}^{\infty} C_m \lambda^{2m-1}.$$

Similarly, from (32), we can have the initial relation

$$B_0^{(1)} = 0, \quad C_0 = 0, \quad A_0^{(1)} - A_0 = 0,$$

and the recursion relation:

$$\begin{aligned} & rB_m^{(1)} - C_m = 0, \quad m \geq 0, \\ & sB_{m+1}^{(1)} + B_m^{(1)} + \left(A_m + A_m^{(1)}\right) = 0, \quad m \geq 0, \\ & s\left(A_{m+1}^{(1)} - A_{m+1}\right) + \left(A_m^{(1)} - A_m\right) + rB_{m+1} - C_{m+1}^{(1)} = 0, \quad m \geq 0. \end{aligned} \tag{33}$$

We choose the initial data satisfying the above initial relation:

$$A_0 = -\frac{1}{2}, \quad B_0 = 0.$$

Then, the stationary discrete zero curvature equation (32) has a unique solution Γ_2 determined by (33). For example, we have

$$\begin{aligned} A_1 &= \frac{r}{ss^{(-1)}}, \quad B_1 = \frac{1}{s^{(-1)}}, \quad C_1 = \frac{r}{s}, \\ A_2 &= -\frac{rr^{(1)}}{s^2 s^{(1)} s^{(-1)}} - \frac{rr^{(-1)}}{s\left(s^{(-1)}\right)^2 s^{(-2)}} - \frac{r^2}{s^2\left(s^{(-1)}\right)^2} - \frac{r}{s^2 s^{(-1)}} - \frac{r}{s\left(s^{(-1)}\right)^2}, \\ B_2 &= -\frac{1}{ss^{(-1)}} - \frac{r}{s\left(s^{(-1)}\right)^2} - \frac{r^{(-1)}}{\left(s^{(-1)}\right)^2 s^{(-2)}}, \\ C_2 &= -\frac{r}{s^2} - \frac{r^2}{s^2 s^{(-1)}} - \frac{rr^{(1)}}{s^2 s^{(1)}}. \end{aligned}$$

Similarly, from (32), we can know (Tu, 1990) that $(E-1)\mathrm{tr}(\Gamma_2^k) = 0$ for all $k \geq 1$. In particular, this tells us that $\mathrm{tr}(\Gamma_2^2) = 2(A^2 + BC)$ is a constant, and let us say γ_2. Then, we obtain a recursion relation for A_m:

$$A_{m+1} = \sum_{i=1}^{m} A_i A_{m-i+1} + \sum_{i=1}^{m+1} B_i B_{m-i+2} - \frac{1}{2}\gamma_2, \quad m \geq 1.$$

This relation, together with the first two recursion relations in (33), implies through the mathematical induction that all lattice functions $A_m, B_m, C_m; m \geq 1$, are local, and they are just difference rational functions in the two dependent variables r and s.

Upon defining

$$W_m = (\lambda^{-2m}\Gamma_2)_- \equiv \begin{pmatrix} \sum_{i=0}^{m} A_i \lambda^{-2m+2i} & \sum_{i=0}^{m} B_i \lambda^{-2m+2i-1} \\ \sum_{i=0}^{m} C_i \lambda^{-2m+2i-1} & -\sum_{i=0}^{m} A_i \lambda^{-2m+2i} \end{pmatrix}, \quad m \geq 0, \quad (34)$$

we can have

$$E(W_m)U - UW_m = \begin{pmatrix} 0 & -sB_{m+1}^{(1)} \\ sC_{m+1} & -\big(A_m^{(1)} - A_m\big)\lambda \end{pmatrix}.$$

To present the associated hierarchy of lattice models, we choose a modification

$$\Theta_m = \begin{pmatrix} S^{(-1)}B_{m+1} + A_m^{(-1)} & 0 \\ 0 & A_m \end{pmatrix},$$

and introduce

$$W^{[m]} = W_m + \Theta_m, \quad m \geq 0. \quad (35)$$

A direct computation leads to

$$\begin{aligned} &\big(EW^{[m]}\big)U - UW^{[m]} \\ &= \begin{pmatrix} 0 & 0 \\ sC_{m+1} + rA_m^{(1)} - rs^{(-1)}B_{m+1} - rA_m^{(-1)} & \dfrac{s\big(A_m^{(1)} - A_m\big)}{\lambda} \end{pmatrix}, \end{aligned}$$

which is consistent with U_{t_m}. Then for all $m \geq 0$, we further introduce the following auxiliary spectral problems associated with the spectral problem (22):

$$\varphi_{t_m} = W^{[m]}\varphi, \quad m \geq 0. \quad (36)$$

Obviously, the compatibility conditions of Eqs. (22) and (36) read as

$$U_{t_m} = \big(EW^{[m]}\big)U - UW^{[m]}, \quad m \geq 0, \quad (37)$$

which give rise to the following hierarchy of lattice models

$$r_{t_m} = sC_{m+1} + rA_m^{(1)} - rs^{(-1)}B_{m+1} - rA_m^{(-1)} = -C_m + rB_m, \quad m \geq 0, \quad (38a)$$

$$s_{t_m} = s\big(A_m^{(1)} - A_m\big) = -\big(A_{m-1}^{(1)} - A_{m-1}\big) - rB_m + C_m^{(1)}, \quad m \geq 0. \quad (38b)$$

These models (38) can be rewritten as

$$u_{t_m} = \begin{pmatrix} r \\ s \end{pmatrix}_{t_m} = J_2 \begin{pmatrix} -\dfrac{A_m}{r} \\ -\dfrac{C_{m+1}}{r} \end{pmatrix}, \quad m \geq 0, \tag{39}$$

where J_2 is defined by (7b). Similarly, we can have the recursion relation

$$\begin{pmatrix} -\dfrac{A_m}{r} \\ -\dfrac{C_{m+1}}{r} \end{pmatrix} = \Psi^{-1} \begin{pmatrix} -\dfrac{A_{m-1}}{r} \\ -\dfrac{C_m}{r} \end{pmatrix}, \quad m \geq 1,$$

which leads to

$$J_2 \begin{pmatrix} -\dfrac{A_m}{r} \\ -\dfrac{C_{m+1}}{r} \end{pmatrix} = J_2 \Psi^{-m} \begin{pmatrix} -\dfrac{A_0}{r} \\ -\dfrac{C_1}{r} \end{pmatrix} = J_2 \Psi^{-m} g_0 = Y_m, \quad m \geq 0,$$

and so the lattice models (39) are just the negative lattice hierarchy (17). This implies that the negative lattice hierarchy (17) is local and it has the discrete zero curvature representations (37). Moreover, the Lax operators $W^{[m]}$ of the negative hierarchy only have negative powers of the spectral parameter λ.

4. CONCLUSIONS AND REMARKS

Two hierarchies of nonlinear bi-Hamiltonian integrable lattice models have been constructed from a difference Hamiltonian operator involving two arbitrary constants. All lattice models in the resulting positive and negative hierarchies have been proved to be local and to possess infinitely many commuting symmetries and infinitely many commuting conserved functionals, which indicates that they are all integrable in the Liouville sense (Tu, 1990). Two examples among the resulting integrable models are

$$r_{t_1} = r\left(s^{(-1)} - s\right) + r\left(r^{(-1)} - r^{(1)}\right), \quad s_{t_1} = rs - r^{(1)}s;$$

and

$$r_{t_1} = \frac{r}{s^{(-1)}} - \frac{r}{s}, \quad s_{t_1} = \frac{r^{(1)}}{s^{(1)}} - \frac{r}{s^{(-1)}}.$$

The first one is from the positive hierarchy; and the second one, from the negative hierarchy. Moreover, a kind of zero curvature representations associated with the discrete spectral problem (22) has been proposed for the two lattice hierarchies. This also provides evidence for integrability of the resulting lattice models by the inverse scattering transfom. The Lax operators for the positive and negative hierarchies (16) and (17) correspond to the positive and negative power expansions with respect to the spectral parameter, respectively. It has also been shown that the integrable lattice models in the positive hierarchy are of polynomial type and the integrable lattice models in the negative hierarchy are of rational type.

Compared with the Ablowitz-Ladik hierarchy (say, see Zeng and Rauch-Wojciechowski, 1995) and the Toda hierarchy (say, see Ma and Fuchssteiner, 1999), the complexity of the lattice hierarchies (16) and (17) should lie between their complexities. Like the Ablowitz-Ladik hierarchy, the discrete spectral problem for (16) and (17) involves the positive and negative powers of the spectral parameter simultaneously, which leads to the existence of two lattice soliton hierarchies. But the lattice hierarchies (16) and (17) have simple bi-Hamiltonian structures like the Toda hierarchy. The Ablowitz-Ladik hierarchy has the higher-degree nonlinearity in the second Hamiltonian operator (see Zeng and Rauch-Wojciechowski, 1995), whose bi-Hamiltonian property seems not to have strictly proved yet.

Other integrable properties of the lattice hierarchies (16) and (17) are interesting as well. Are there any Bäcklund transformation and soliton solutions? What are master symmetries and π-functions? It is particularly interesting to find soliton, positon, negaton, and complexiton solutions to the above two typical integrable models. The resulting Hamiltonian operator (6) may also contain other hierarchies of integrable lattice models. The arbitrariness of two constants brings choices to present integrable lattice models. The higher-order matrix generalization of the Hamiltonian operator (6) and the combination of the Hamiltonian operator (6) with constant coefficient matrix operators must be good candidates which lead to different Hamiltonian pairs in constructing integrable lattice models (see Tu and Ma, 1992, and Ma, 1990, for examples in the continuous case). We hope that there will be answers to these questions and we love to have more fishes.

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APPENDIX. THE PROOF OF THE JACOBI IDENTITY

We would like to give a concrete checking of the Jacobi identity

$$\langle J'(u)[Jf]g, h\rangle + \mathrm{Cycle}(f, g, h) = 0$$

for the Hamiltonian operator defined by (6). Assume that

$$f = (f_1(n,t),\ f_2(n,t))^{\mathrm{T}}, \quad g = (g_1(n,t), g_2(n,t))^{\mathrm{T}}, \quad h = (h_1(n,t), h_2(n,t))^{\mathrm{T}}$$

are three arbitrary functions, which are required to be rapidly vanishing at the infinity. We combine terms on the left side of the Jacobi identity, i.e., in $\langle J'(u)[Jf]g, h\rangle +$ Cycle(f, g, h), containing α^2, β^2, and $\alpha\beta$, respectively. Through a laborious but straightforward computation, we can find that the coefficients of α^2, β^2, and $\alpha\beta$ read as

$$\begin{aligned}
&\sum_{n\in Z}[rf_2g_2h_1 - r(E^{-1}f_2)g_2h_1 - rf_2(E^{-1}g_2)h_1 + r(E^{-1}f_2)(E^{-1}g_2)h_1\\
&\quad - rf_2g_1h_2 + (Er)(Ef_2)(Eg_1)h_2 + r(E^{-1}f_2)g_1h_2 - (Er)f_2(Eg_1)h_2\\
&\quad + rf_2(E^{-1}g_2)h_2 - r(E^{-1}f_2)(E^{-1}g_2)h_2 - (Er)(Ef_2)(Eg_2)h_2\\
&\quad + (Er)f_2(Eg_2)h_2](n) + \text{Cycle}(f, g, h),
\end{aligned}$$

$$\begin{aligned}
&\sum_{n\in Z}[(r(Er)^2(Ef_1)(Eg_1)(Eh_1) - r(E^{-1}r)(Er)(E^{-1}f_1)(Eg_1)(Eh_1)\\
&\quad + r(Er)sf_2(Eg_1)(Eh_1) - r(Er)(E^{-1}s)(E^{-1}f_2)(Eg_1)(Eh_1)\\
&\quad + r(Er)(E^2r)(E^2f_1)(Eg_1)(Eh_1) + r(Er)(Es)(Ef_2)(Eg_1)(Eh_1)\\
&\quad - r^2(Er)f_1(Eg_1)(Eh_1) - r(Er)sf_2(Eg_1)(Eh_1) - r^2(E^{-1}r)f_1(E^{-1}g_1)\\
&\quad \times (E^{-1}h_1) + r(E^{-1}r)(E^{-2}r)(E^{-2}f_1)(E^{-1}g_1)(E^{-1}h_1)\\
&\quad - r(E^{-1}r)(E^{-1}s)(E^{-1}f_2)(E^{-1}g_1)(E^{-1}h_1) + r(E^{-1}r)(E^{-2}s)(E^{-2}f_2)\\
&\quad \times (E^{-1}g_1)(E^{-1}h_1) - r(Er)(E^{-1}r)(Ef_1)(E^{-1}g_1)(E^{-1}h_1) + r(E^{-1}r)^2\\
&\quad \times (E^{-1}f_1)(E^{-1}g_1)(E^{-1}h_1) - r(E^{-1}r)sf_2(E^{-1}g_1)(E^{-1}h_1) + r(E^{-1}r)(E^{-1}s)\\
&\quad \times (E^{-1}f_2)(E^{-1}g_1)(E^{-1}h_1) - r^2sf_1g_2h_1 + r(Er)s(Ef_1)g_2h_1\\
&\quad + r(Er)s(Ef_1)g_2h_1 - r(E^{-1}r)s(E^{-1}f_1)g_2h_1 + rs^2f_2g_2h_1\\
&\quad - rs(E^{-1}s)(E^{-1}f_2)g_2h_1 - r(Er)(E^{-1}s)(Ef_1)(E^{-1}g_2)(E^{-1}h_1)\\
&\quad + r(E^{-1}r)(E^{-1}s)(E^{-1}f_1)(E^{-1}g_2)(E^{-1}h_1) - rs(E^{-1}s)f_2(E^{-1}g_2)(E^{-1}h_1)\\
&\quad + r(E^{-1}s)^2(E^{-1}f_2)(E^{-1}g_2)(E^{-1}h_1) + r(E^{-1}r)(E^{-1}s)(E^{-1}f_1)\\
&\quad \times (E^{-1}g_2)(E^{-1}h_1) - r^2(E^{-1}s)f_1(E^{-1}g_2)(E^{-1}h_1) - r(Er)s(Ef_1)g_1h_2\\
&\quad + r(E^{-1}r)s(E^{-1}f_1)g_1h_2 - rs^2f_2g_1h_2 + rs(E^{-1}s)(E^{-1}f_2)g_1h_2 + r^2sf_1g_1h_2\\
&\quad - r(Er)s(Ef_1)g_1h_2 + (Er)(E^2r)s(E^2f_1)(Eg_1)(Eh_2) - r(Er)sf_1(Eg_1)(Eh_2)\\
&\quad + (Er)s(Es)(Ef_2)(Eg_1)(Eh_2) - (Er)s^2f_2(Eg_1)(Eh_2)\\
&\quad - r(Er)sf_1(Eg_1)(Eh_2) + (Er)^2s(Ef_1)(Eg_1)(Eh_2)](n) + \text{Cycle}\,(f, g, h),
\end{aligned}$$

and

$$
\begin{aligned}
\sum_{n\in Z}&[r(Er)(E^{-1}f_2)(Eg_1)h_1 - r(Er)f_2(Eg_1)h_1 - r(Er)(Ef_2)(Eg_1)h_1\\
&+r(Er)f_2(Eg_1)h_1 + r(E^{-1}r)f_2(E^{-1}g_1)h_1 - r(E^{-1}r)(E^{-1}f_2)(E^{-1}g_1)h_1\\
&-r(E^{-1}r)(E^{-1}f_2)(E^{-1}g_1)h_1 - r(E^{-1}r)(E^{-2}f_2)(E^{-1}g_1)h_1\\
&-r(Er)(Ef_1)g_2h_1 + r(E^{-1}r)(E^{-1}f_1)g_2h_1 - rsf_2g_2h_1\\
&+r(E^{-1}s)(E^{-1}f_2)g_2h_1 - rsf_2g_2h_1 + rs(E^{-1}f_2)g_2h_1 + r^2f_1g_2h_1\\
&-r(Er)(Ef_1)g_2h_1 - r^2(E^{-1}f_2)g_2h_1 + r(Er)(Ef_2)g_2h_1\\
&+r(Er)(Ef_1)(E^{-1}g_2)h_1 + rsf_2(E^{-1}g_2)h_1 - r(E^{-1}r)(E^{-1}f_1)(E^{-1}g_2)h_1\\
&-r(E^{-1}s)(E^{-1}f_2)(E^{-1}g_2)h_1 + r(E^{-1}s)f_2(E^{-1}g_2)h_1 - r(E^{-1}s)\\
&\times(E^{-1}f_2)(E^{-1}g_2)h_1 - r(E^{-1}r)(E^{-1}f_1)(E^{-1}g_2)h_1 + r^2f_1(E^{-1}g_2)h_1\\
&+r(E^{-1}r)(E^{-2}f_2)(E^{-1}g_2)h_1 - r^2f_2(E^{-1}g_2)h_1 + r(Er)(Ef_1)g_1h_2\\
&-r(E^{-1}r)(E^{-1}f_1)g_1h_2 + rsf_2g_1h_2 - r(E^{-1}s)(E^{-1}f_2)g_1h_2 + rsf_2g_1h_2\\
&-rs(E^{-1}f_2)g_1h_2 - r^2f_1g_1h_2 + r(Er)(Ef_1)g_1h_2 + r^2(E^{-1}f_2)g_1h_2\\
&-r(Er)(Ef_2)g_1h_2 - (Er)(Er)^2(E^2f_1)(Eg_1)h_2 + r(Er)f_1(Eg_1)h_2\\
&-(Er)(Es)(Ef_2)(Eg_1)h_2 + (Er)sf_2(Eg_1)h_2 - (Er)s(Ef_2)(Eg_1)h_2\\
&+(Er)sf_2(Eg_1)h_2 + r(Er)f_1(Eg_1)h_2 - (Er)^2(Ef_1)(Eg_1)h_2\\
&-r(Er)(E^{-1}f_2)(Eg_1)h_2 + (Er)^2(Ef_2)(Eg_1)h_2 - r(Er)(Ef_1)(E^{-1}g_2)h_2\\
&+r(E^{-1}r)(E^{-1}f_1)(E^{-1}g_2)h_2 - rsf_2(E^{-1}g_2)h_2\\
&+r(E^{-1}s)(E^{-1}f_2)(E^{-1}g_2)h_2 + (Er)(E^2r)(E^2f_1)(Eg_2)h_2\\
&-r(Er)f_1(Eg_2)h_2 + (Er)(Es)(Ef_2)(Eg_2)h_2 - (Er)sf_2(Eg_2)h_2](n)\\
&+\mathrm{Cycle}\,(f,g,h),
\end{aligned}
$$

respectively. By a careful checking, we see that these three sums are all equal to zero. Therefore, the Jacobi identity for the Hamiltonian operator (6) holds.

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